Rational operations; a philosophical approach

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Addition of two numbers may intuitively be interpreted as taking to line segments, placing them after another, and measuring the length of both of them. Let us have the number 2, for example, and also the number 3. Let us interpret them as centimeters on a ruler. First, we draw a 2 centimeter line. Then we draw a new line, starting right where the first one ended. Now we use the ruler to measure the length of the new line. Not surprisingly, the new line has a length of 5. More formally, adding to numbers a and b is the same as adding to line segments and measure the new line. It is easily seen that the fundamental properties of addition are still valid (I'm thinking of the fact that a+b=b+a and that (a+b)+c=a+(b+c)).

Now, what about subtraction? Essentialy, what we do is the same. To perform the calculation b-a, we first draw a line of length b, then a line with length a, starting where b ended, but now we draw it in the opposite direction. Where the new line ends, we draw a mark and measure the length from the start of b to the mark. This is the answer of b-a. A problem does however occur when a is greater than b. This is where negative numbers are needed. Introducing them is easy: just mark the point where the first line is drawn and call this point "zero"(0). If a is greater than b, the answer is simply the length from the zero point to the endpoint of the second line.

So what about multiplication? Unfortunaly, this analogy (number vs line segments) does not fully show us all properties of the numbers. Using this analogy, we must limit ourselves to multiplying reals and integers (at best!). Now, what is multiplication? Easy: first, find your first number and call it a. This number may perfectly well be an irrational if you prefer so, or an integer if you like them better. Anyways, draw this number with your ruler. Then, choose an integer (here you have no choice!) and name it n. Then - remembering you have already drawn 'a' one time - draw your a n-1 times more succesively on your sheet of paper. Measure the length of this new long line, and you've just calculated a*n. The fundamental properties of multiplication are obvious: ab=ba and a(bc)=ab(c). If you do this a few times, you will also realize that the distributive is true: a(b+c)=ab+ac.

Now, what about multiplication of negative numbers? Let us do this in two steps: First, we introduce the notion of multiplying a number by -1. Using our analogy, this is equivalent to first draw a line, then mirrroring this line about the zero point. The answer of -1*a is now, obviously, the length of a with a minus sign put first. Therefore we say that -1*a=-a. Using this analogy, it is quite intuitively obvious that -(-a)=a. This corresponds to drawing a negative line, and then mirroring it - the answer is obviously positive.

Now, let us multiply a positive and a negative number. Say a*(-b). First, we see that that (-b)=-1*b. This implies that a*(-b) may also be written as -1*(ab), which is a operation already defined.